

ON A CONSISTENT ESTIMATOR OF A USEFUL SIGNAL IN ORNSTEIN-UHLENBECK STOCHASTIC MODEL IN $\mathbb{C}[-l, l[$

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ABSTRACT. It is considered a transmission process of a useful signal in Ornstein-Uhlenbeck stochastic model in $\mathbb{C}[-l, l[$ defined by the stochastic differential equation

$$d\Psi(t, x, \omega) = \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n} \Psi(t, x, \omega) dt + \sigma dW(t, \omega)$$

with initial condition

$$\Psi(0, x, \omega) = \Psi_0(x) \in FD^{(0)}[-l, l[,$$

where $m \geq 1$, $(A_n)_{0 \leq n \leq 2m} \in \mathbb{R}^+ \times \mathbb{R}^{2m-1}$, $((t, x, \omega) \in [0, +\infty[\times [-l, l[\times \Omega)$, $\sigma \in \mathbb{R}^+$, $\mathbb{C}[-l, l[$ is Banach space of all real-valued bounded continuous functions on $[-l, l[$, $FD^{(0)}[-l, l[\subset \mathbb{C}[-l, l[$ is class of all real-valued bounded continuous functions on $[-l, l[$ whose Fourier series converges to himself everywhere on $[-l, l[$, $(W(t, \omega))_{t \geq 0}$ is a Wiener process and $\Psi_0(x)$ is a useful signal.

By use a sequence of transformed signals $(Z_k)_{k \in N} = (\Psi(t_0, x, \omega_k))_{k \in N}$ at moment $t_0 > 0$, consistent and infinite-sample consistent estimates of the useful signal Ψ_0 is constructed under assumption that parameters $(A_n)_{0 \leq n \leq 2m}$ and σ are known. Animation and simulation of the Ornstein-Uhlenbeck process in Banach space $\mathbb{C}[-l, l[$ and results of calculations of estimates of a useful signal in the same stochastic model are also presented.

1. INTRODUCTION

Suppose that Θ is a vector subspace of the Banach space $\mathbb{C}[-l, l[$ equipped with usual norm, where $\mathbb{C}[-l, l[$ denotes the class of all bounded continuous functions on $[-l, l[$.

In the information transmitting theory we consider Ornstein-Uhlenbeck stochastic system

$$\begin{aligned} \xi(t, x, \omega)_{x \in [-l, l[} &= e^{t \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}} \theta(x)_{x \in [-l, l[} + \\ &\sigma \int_0^t e^{(t-\tau) \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}} \times I_{[-l, l[}(x) dW(\tau, \omega), \end{aligned} \quad (1.1)$$

where $(A_n)_{0 \leq n \leq 2m} \in \mathbb{R}^+ \times \mathbb{R}^{2m-1}$ ($m \geq 1$), $\theta \in \Theta$ is a useful signal, $(W(t, \cdot))_{t \geq 0}$ is Winner processes (the so-called “white noises”) defined on the probability space (Ω, \mathbf{F}, P) , $(\xi(t, \cdot, \omega))$ (equivalently, $\xi(t, x, \omega)_{x \in [-l, l[}$) is a transformed signal for $(t, \omega) \in [0, +\infty[\times \Omega$, $I_{[-l, l[}$ denotes the indicator function of the interval $[-l, l[$.

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Let μ be a Borel probability measure on $\mathbb{C}^{[0,+\infty[[-l, l[$ defined by generalized “white noise”

$$\left(\sigma \int_0^t e^{(t-\tau) \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}} \times I_{[-l, l[} dW(\tau, \omega) \right)_{t \geq 0}. \quad (1.2)$$

Then we have

$$\begin{aligned} & (\forall X)(X \in \mathbf{B}(\mathbb{C}^{[0,+\infty[[-l, l[) \rightarrow \mu(X) = P(\{\omega : \omega \in \Omega \\ & \& (\sigma \int_0^t e^{(t-\tau) \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}} \times I_{[-l, l[} dW_k(\tau, \omega))_{t \geq 0} \in X\})), \end{aligned} \quad (1.3)$$

where $\mathbf{B}(\mathbb{C}^{[0,+\infty[[-l, l[)$ is the Borel σ -algebra of subsets of the space $\mathbb{C}^{[0,+\infty[[-l, l[$.

Let λ be a Borel probability measure on $\mathbb{C}^{[0,+\infty[[-l, l[$ defined by transformed signal $\xi(t, x, \omega)$ that is

$$\begin{aligned} & (\forall X)(X \in \mathbf{B}(\mathbb{C}^{[0,+\infty[[-l, l[) \rightarrow \mu(X) = P(\{\omega : \omega \in \Omega \\ & \& (\xi(t, x, \omega))_{t \geq 0} \in X\})), \end{aligned} \quad (1.4)$$

In the information transmitting theory, the general decision is that the Borel probability measure λ , defined by the transformed signal coincide with $e^{t \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n} \theta}$ shift μ_θ of the measure μ for some $\theta_0 \in \Theta$ provided that

$$(\exists \theta_0)(\theta_0 \in \Theta \rightarrow (\forall X)(X \in \mathbf{B}(\mathbb{C}^{[0,+\infty[[-l, l[) \rightarrow \lambda(X) = \mu_{\theta_0}(X))), \quad (1.5)$$

where $\mu_\theta(\cdot) = \mu(\cdot + (e^{t \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n} \theta})_{t \geq 0})$.

Here we consider a particular case of the above model when a vector space of useful signals Θ coincides with $FD^0[-l, l[$, where $FD^0[-l, l[\subset \mathbb{C}[-l, l[$ denotes a vector space of all bounded continuous real-valued functions on $[-l, l[$ whose Fourier series converges to himself everywhere on $[-l, l[$.

Definition 1.1. Following [11], a triplet

$$((\mathbb{C}^{[0,+\infty[[-l, l[})^{\mathbb{N}}, \mathbf{B}((\mathbb{C}^{[0,+\infty[[-l, l[})^{\mathbb{N}}), \mu_\theta^{\mathbb{N}})_{\theta \in \Theta} \quad (1.6)$$

is called a statistical structure described the stochastic system (1.1).

Definition 1.2. Following [11], a Borel measurable function $T_n : (\mathbb{C}^{[0,+\infty[[-l, l[})^{\mathbb{N}} \rightarrow \Theta$ ($n \in \mathbb{N}$) is called a consistent estimate of a parameter θ for the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \Theta}$ if the condition

$$\mu_\theta^{\mathbb{N}}(\{(x_k)_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in (\mathbb{C}^{[0,+\infty[[-l, l[})^{\mathbb{N}} \& \lim_{n \rightarrow \infty} \|T_n(x_1, \dots, x_n) - \theta\| = 0\}) = 1 \quad (1.7)$$

holds for each $\theta \in \Theta$, where $\|\cdot\|$ is a usual norm in $\mathbb{C}[-l, l[$.

Definition 1.3. Following [11], a Borel measurable function $T : (\mathbb{C}^{[0,+\infty[[-l, l[})^{\mathbb{N}} \rightarrow \Theta$ is called an infinite-sample consistent estimate of a parameter θ for the family $(\mu_\theta^{\mathbb{N}})_{\theta \in \Theta}$ if the condition

$$\mu_\theta^{\mathbb{N}}(\{(x_k)_{k \in \mathbb{N}} : (x_k)_{k \in \mathbb{N}} \in (\mathbb{C}^{[0,+\infty[[-l, l[})^{\mathbb{N}} \& T((x_k)_{k \in \mathbb{N}}) = \theta\}) = 1 \quad (1.8)$$

holds for each $\theta \in \Theta$.

The main goal of the present paper is construct consistent and infinite-sample estimators of the useful signal for the stochastic model (1.1) which is a particular case of the Ornstein-Uhlenbeck process in $\mathbb{C}[-l, l[$. Concerning estimations of parameters for another versions of the Ornstein-Uhlenbeck processes the reader can consult with [2], [7], [5], [4].

The rest of the present paper is the following.

Section 2 contains some auxiliary notions and fact from theories of ordinary and stochastic differential equations.

In Section 3 we present our main results.

In Section 4 we present animations and simulations of the Ornstein-Uhlenbeck process in $\mathbb{C}[-l, l[$ and present results of calculations of the estimator of a useful signal when parameters $(A_n)_{0 \leq n \leq 2m}, \sigma$ and a sample of transformed signals at moment $t_0 > 0$ defined by (1.1) are known.

In Section 5 we consider discussion and conclusion.

2. MATERIALS AND METHODS

We begin this section by a short description of a certain result concerning a solution of some differential equations with initial value problem obtained in the paper [6]. Further, by use this approach and technique developed in [8], their some applications for a solution of the Ornstein-Uhlenbeck stochastic differential equation in $\mathbb{C}[-l, l[$ are obtained. At end of this section, well known Kolmogorov Strong Law of Large Numbers is presented.

Lemma 2.1 ([6], Corollary 2.1, p. 6). *For $m \geq 1$, let us consider a linear partial differential equation*

$$\frac{\partial}{\partial t} \Psi(t, x) = \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n} \Psi(t, x) \quad ((t, x) \in [0, +\infty[\times [-l, l]) \quad (2.1)$$

with initial condition

$$\Psi(0, x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\left(\frac{k\pi x}{l}\right) + d_k \sin\left(\frac{k\pi x}{l}\right) \in FD^{(0)}[-l, l[. \quad (2.2)$$

If $(\frac{c_0}{2}, c_1, d_1, c_2, d_2, \dots)$ is such a sequence of real numbers that a series $\Psi(t, x)$ defined by

$$\begin{aligned} \Psi(t, x) = & \frac{e^{tA_0} c_0}{2} + \sum_{k=1}^{\infty} e^{\sigma_k t} \left((c_k \cos(\omega_k t) + \right. \\ & \left. d_k \sin(\omega_k t)) \cos\left(\frac{k\pi x}{l}\right) + (d_k \cos(\omega_k t) - c_k \sin(\omega_k t)) \sin\left(\frac{k\pi x}{l}\right) \right) \end{aligned} \quad (2.3)$$

belongs to the class $FD^{(2m)}[-l, l[$ as a series of a variable x for all $t \geq 0$, and is differentiable term by term as a series of a variable t for all $x \in [-l, l[$, then Ψ is a solution of (2.1)-(2.2).

By use an approach developed in [8], we get the validity of the following assertion.

Lemma 2.2. *For $m \geq 1$, let us consider Ornstein-Uhlenbeck process in $\mathbb{C}[-l, l[$ defined by the stochastic differential equation*

$$d\Psi(t, x, \omega) = \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n} \Psi(t, x, \omega) dt + \sigma dW(t, \omega) I_{[-l, l[}(x) \quad ((t, x, \omega) \in [0, +\infty[\times [-l, l[\times \Omega) \quad (2.4)$$

with initial condition

$$\Psi(0, x, \omega) = \Psi_0(x), \quad (2.5)$$

where $(A_n)_{0 \leq n \leq 2m} \in \mathbb{R}^+ \times \mathbb{R}^{2m-1}$, $(W(t, \omega))_{t \geq 0}$ is a Wiener process and

$$\Psi_0(x) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\left(\frac{k\pi x}{l}\right) + d_k \sin\left(\frac{k\pi x}{l}\right) \in FD^{(0)}[-l, l]. \quad (2.6)$$

If $(\frac{c_0}{2}, c_1, d_1, c_2, d_2, \dots)$ is such a sequence of real numbers that a series

$$\begin{aligned} & \frac{e^{tA_0} c_0}{2} + \sum_{k=1}^{\infty} e^{\sigma_k t} \left((c_k \cos(\omega_k t) + \right. \\ & \left. d_k \sin(\omega_k t)) \cos\left(\frac{k\pi x}{l}\right) + (d_k \cos(\omega_k t) - c_k \sin(\omega_k t)) \sin\left(\frac{k\pi x}{l}\right) \right) + \sigma \int_0^t e^{(t-\tau)A_0} dW_\tau(\omega) I_{[-l, l[}(x) \end{aligned} \quad (2.7)$$

belongs to the class $FD^{(2m)}[-l, l[$ as a series of a variable x for all $t \geq 0, \omega \in \Omega$, and is differentiable term by term as a series of a variable t for all $x \in [-l, l[, \omega \in \Omega$, then the solution of (2.5)-(2.6) is given by

$$\Psi(t, x, \omega) = e^{t \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}} (\Psi(0, x, \omega)) + \sigma \int_0^t e^{(t-s) \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}} dW(s, \omega).$$

Proof. Putting $\mathbb{A} = \sum_{n=0}^{2m} A_n \partial^n / \partial x^n$ and $f(t, \Psi(t, x, \omega)) = e^{-t\mathbb{A}} \Psi(t, x, \omega)$, we get

$$\begin{aligned} df(t, \Psi(t, x, \omega)) &= -\mathbb{A} e^{-t\mathbb{A}} \Psi(t, x, \omega) dt + e^{-t\mathbb{A}} d\Psi(t, x, \omega) \\ &= -\mathbb{A} e^{-t\mathbb{A}} \Psi(t, x, \omega) dt + e^{-t\mathbb{A}} (\mathbb{A} \Psi(t, x, \omega) + \sigma dW(t, \omega)) = \sigma e^{-t\mathbb{A}} dW(t, \omega). \end{aligned}$$

By integration of both sides we get

$$f(t, \Psi(t, x, \omega)) - f(0, \Psi(0, x, \omega)) = \sigma \int_0^t e^{-\tau\mathbb{A}} dW(\tau, \omega)$$

which implies

$$e^{-t\mathbb{A}} \Psi(t, x, \omega) - e^{-0\mathbb{A}} \Psi(0, x, \omega) = \sigma \int_0^t e^{-\tau\mathbb{A}} dW(\tau, \omega).$$

Now we get

$$e^{t\mathbb{A}} (e^{-t\mathbb{A}} (\Psi(t, x, \omega)) - e^{t\mathbb{A}} (e^{-0\mathbb{A}} (\Psi(0, x, \omega)))) = e^{t\mathbb{A}} (\sigma \int_0^t e^{-\tau\mathbb{A}} dW(\tau, \omega)),$$

which is equivalent to the equality

$$\Psi(t, x, \omega) = e^{t\mathbb{A}} (\Psi(0, x, \omega)) + \sigma \int_0^t e^{(t-\tau)\mathbb{A}} I_{[-l, l[}(x) dW(\tau, \omega).$$

□

Remark 2.3. Under condition of Lemma 2.2 we have

$$\begin{aligned} \Psi(t, x, \omega) &= \frac{e^{tA_0} c_0}{2} + \sum_{k=1}^{\infty} e^{\sigma_k t} \left((c_k \cos(\omega_k t) + \right. \\ &\quad \left. d_k \sin(\omega_k t)) \cos\left(\frac{k\pi x}{l}\right) + (d_k \cos(\omega_k t) - c_k \sin(\omega_k t)) \sin\left(\frac{k\pi x}{l}\right) \right) + \\ &\quad \sigma \int_0^t e^{(t-\tau)A_0} I_{[-l, l]}(x) dW(\tau, \omega). \end{aligned} \quad (2.8)$$

Lemma 2.4. *Under conditions of Lemma 2.2, the following conditions are valid:*

- (i) $E\Psi(t, x, \cdot) = \frac{e^{tA_0} c_0}{2} + \sum_{k=1}^{\infty} e^{\sigma_k t} \left((c_k \cos(\omega_k t) + d_k \sin(\omega_k t)) \cos\left(\frac{k\pi x}{l}\right) + (d_k \cos(\omega_k t) - c_k \sin(\omega_k t)) \sin\left(\frac{k\pi x}{l}\right) \right);$
- (ii) $\text{cov}(\Psi(s, x, \cdot), \Psi(t, x, \cdot)) = \frac{\sigma^2}{2A_0} (e^{-A_0(t-s)} - e^{-A_0(t+s)});$
- (iii) $\text{var}(\Psi(s, x, \cdot)) = \frac{\sigma^2}{2A_0} (1 - e^{-2A_0 s});$

Proof. The validity of the item (i) is obvious. In order to prove the validity of the items (ii)-(iii), we can use the Ito isometry to calculate the covariance function by

$$\begin{aligned} \text{cov}(\Psi(s, x, \cdot), \Psi(t, x, \cdot)) &= E[(\Psi(s, x, \cdot) - E[\Psi(s, x, \cdot)])(\Psi(t, x, \cdot) - E[\Psi(t, x, \cdot)])] \\ &= E \left[\int_0^s \sigma e^{A_0(u-s)} dW(u, \omega) \int_0^t \sigma e^{A_0(v-t)} dW(v, \omega) \right] \\ &= \sigma^2 e^{-A_0(s+t)} E \left[\int_0^s e^{A_0 u} dW(u, \omega) \int_0^t e^{A_0 v} dW(v, \omega) \right] \\ &= \frac{\sigma^2}{2A_0} e^{-A_0(s+t)} (e^{2A_0 \min(s, t)} - 1). \end{aligned}$$

Thus if $s < t$ (so that $\min(s, t) = s$), then we have

$$\text{cov}(\Psi(s, x, \cdot), \Psi(t, x, \cdot)) = \frac{\sigma^2}{2A_0} (e^{-A_0(t-s)} - e^{-A_0(t+s)}).$$

Similarly, if $s = t$ (so that $\min(s, t) = s$), then we have

$$\text{var}(\Psi(s, x, \cdot)) = \frac{\sigma^2}{2A_0} (1 - e^{-2A_0 s}).$$

□

In the next section we will need the well known fact from the probability theory (see, for example, [9], p. 390).

Lemma 2.5. *(Kolmogorov's strong law of large numbers) Let X_1, X_2, \dots be a sequence of independent identically distributed random variables defined on the probability space (Ω, \mathcal{F}, P) . If these random variables have a finite expectation m (i.e., $E(X_1) = E(X_2) = \dots = m < \infty$), then the following condition*

$$P(\{\omega : \lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n X_k(\omega) = m\}) = 1 \quad (2.9)$$

holds true.

3. RESULTS

In this section, by the use of Kolmogorov Strong Law of Large Numbers we construct a consistent and an infinite-sample consistent estimators of a useful signal which is transmitted by the Ornstein-Uhlenbeck stochastic system (1.1).

Theorem 3.1. *Let consider $\mathbb{C}[-l, l[$ -valued stochastic process $(\xi(t, x, \omega))_{x \in [-l, l[}, t \geq 0$ defined by*

$$\xi(t, x, \omega)_{x \in [-l, l[} = e^{t\mathbb{A}}(\theta(x)_{x \in [-l, l[}) + \sigma \int_0^t e^{(t-\tau)\mathbb{A}} I_{[-l, l[}(x)_{x \in [-l, l[} dW(\tau, \omega), \quad (3.1)$$

where $\theta \in FD^{(0)}$ and $\mathbb{A} = \sum_{n=0}^{2m} A_n \frac{\partial^n}{\partial x^n}$. Assume that all conditions of Lemma 2.9 are satisfied. For a fixed $t = t_0 > 0$, we denote by μ_θ a probability measure in $\mathbb{C}[-l, l[$ defined by the random element $(\Xi(t_0, \omega))$. For each $(Z_k)_{k \in \mathbb{N}} \in (\mathbb{C}[-l, l[)^{\mathbb{N}}$ we put

$$T_n((Z_k)_{k \in \mathbb{N}}) = e^{-t_0 \mathbb{A}} \left(\frac{\sum_{k=1}^n Z_k}{n} \right). \quad (3.2)$$

Then T_n is a consistent estimate of a useful signal θ provided that

$$\mu_\theta^{\mathbb{N}} \{ (Z_k)_{k \in \mathbb{N}} : \lim_{n \rightarrow \infty} \|T_n((Z_k)_{k \in \mathbb{N}}) - \theta\| = 0 \} = 1 \quad (3.3)$$

for each $\theta \in FD^{(0)}[-l, l[$.

Proof. For each $X \in B(\mathbb{C}[-l, l[)$ we have

$$\begin{aligned} \mu_\theta(X) &= P\{\omega : \Xi(t_0, \omega) \in X\} = P\{\omega : e^{t_0 \mathbb{A}}(\theta) + \sigma \int_0^{t_0} e^{(t_0-\tau)\mathbb{A}} I_{[-l, l[} dW(\tau, \omega) \in X\} = \\ &= P\{\omega : \sigma \int_0^{t_0} e^{(t_0-\tau)\mathbb{A}} dW(\tau, \omega) I_{[-l, l[} \in (X - e^{t_0 \mathbb{A}}(\theta)) \cap \{\alpha I_{[-l, l[} : \alpha \in \mathbb{R}\}\}. \end{aligned}$$

We have

$$\begin{aligned} &\mu_\theta^{\mathbb{N}} \{ (Z_k)_{k \in \mathbb{N}} : \lim_{n \rightarrow \infty} \|T_n((Z_k)_{k \in \mathbb{N}}) - \theta\| = 0 \} = \\ &\mu_\theta^{\mathbb{N}} \{ (Z_k)_{k \in \mathbb{N}} : \lim_{n \rightarrow \infty} \|e^{-t_0 \mathbb{A}} \left(\frac{\sum_{k=1}^n Z_k}{n} \right) - \theta\| = 0 \} = \\ &\mu_\theta^{\mathbb{N}} \{ (Z_k)_{k \in \mathbb{N}} : \lim_{n \rightarrow \infty} \left\| \frac{\sum_{k=1}^n Z_k}{n} - e^{t_0 \mathbb{A}}(\theta) \right\| = 0 \} = \\ &\mu_\theta^{\mathbb{N}} \{ (Z_k)_{k \in \mathbb{N}} : Z_k \in \{\alpha_k I_{[-l, l[} : \alpha_k \in \mathbb{R}\} + e^{t_0 \mathbb{A}}(\theta) \text{ \& } \lim_{n \rightarrow \infty} \left\| \frac{\sum_{k=1}^n Z_k}{n} - e^{t_0 \mathbb{A}}(\theta) \right\| = 0 \} = \\ &\mu_\theta^{\mathbb{N}} \{ (Z_k)_{k \in \mathbb{N}} : (\exists (\beta_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty) (Z_k = \beta_k I_{[-l, l[} + \\ &\quad e^{t_0 \mathbb{A}}(\theta) \text{ \& } \lim_{n \rightarrow \infty} \left\| \frac{\sum_{k=1}^n Z_k}{n} - e^{t_0 \mathbb{A}}(\theta) \right\| \\ &= 0 \} = \mu_\theta^{\mathbb{N}} \{ (Z_k)_{k \in \mathbb{N}} : (\exists (\beta_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty) (Z_k = \beta_k I_{[-l, l[} + \\ &\quad e^{t_0 \mathbb{A}}(\theta) \text{ \& } \lim_{n \rightarrow \infty} \left\| \frac{\sum_{k=1}^n \beta_k}{n} I_{[-l, l[} \right\| \\ &= 0 \} = \mu_\theta^{\mathbb{N}} \{ (Z_k)_{k \in \mathbb{N}} : (\exists (\beta_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty) (Z_k = \beta_k I_{[-l, l[} + \\ &\quad e^{t_0 \mathbb{A}}(\theta) \text{ \& } \lim_{n \rightarrow \infty} \left| \frac{\sum_{k=1}^n \beta_k}{n} \right| = 0 \} = \end{aligned}$$

$$\gamma_{(0,s)}^{\mathbb{N}}\{\exists(\beta_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty : \lim_{n \rightarrow \infty} \left| \frac{\sum_{k=1}^n \beta_k}{n} \right| = 0\} = 1,$$

where $\gamma_{(0,s)}$ denotes the Gaussian measure in \mathbb{R} with the mean 0 and the variance $s^2 = \frac{\sigma^2}{2A_0}(1 - e^{-2A_0 t_0})$. The validity of the last equality is a direct consequence of Lemmas 2.9-2.10. \square

Theorem 3.2. (Continue) Let $\theta^* \in FD^{(0)}$. For $(Z_k)_{k \in \mathbb{N}} \in (\mathbb{C}[-l, l])^{\mathbb{N}}$ we put $T((Z_k)_{k \in \mathbb{N}}) = \lim_{n \rightarrow \infty} T_n((Z_k)_{k \in \mathbb{N}})$, if the sequence $(T_n((Z_k)_{k \in \mathbb{N}}))_{n \rightarrow \mathbb{N}}$ is convergent and this limit belongs to the class $FD^{(0)}$, and $T((Z_k)_{k \in \mathbb{N}}) = \theta^*$, otherwise. Then $T : (\mathbb{C}[-l, l])^{\mathbb{N}} \rightarrow FD^{(0)}$ is an infinite-sample consistent estimate of a useful signal $\theta \in FD^{(0)}$ with respect to family $\mu_\theta^{\mathbb{N}}$ provided that the condition

$$\mu_\theta^{\mathbb{N}}(\{(Z_k)_{k \in \mathbb{N}} : (Z_k)_{k \in \mathbb{N}} \in (\mathbb{C}^{[0, +\infty[[-l, l])^{\mathbb{N}}} \& T((Z_k)_{k \in \mathbb{N}}) = \theta\}) = 1 \quad (3.4)$$

holds for each $\theta \in FD^0[-l, l[$.

Proof. For $\theta \in FD^0[-l, l[$, by the use the result of Theorem 3.1 we get

$$\begin{aligned} & \mu_\theta^{\mathbb{N}}(\{(Z_k)_{k \in \mathbb{N}} : (Z_k)_{k \in \mathbb{N}} \in (\mathbb{C}^{[0, +\infty[[-l, l])^{\mathbb{N}}} \& T((Z_k)_{k \in \mathbb{N}}) = \theta\}) \geq \\ & \mu_\theta^{\mathbb{N}}(\{(Z_k)_{k \in \mathbb{N}} : (Z_k)_{k \in \mathbb{N}} \in (\mathbb{C}^{[0, +\infty[[-l, l])^{\mathbb{N}}} \& \lim_{n \rightarrow \infty} T_n((Z_k)_{k \in \mathbb{N}}) = \theta\}) = 1. \end{aligned}$$

\square

4. ANIMATION AND SIMULATION OF THE ORNSTEIN-UHLENBECK PROCESS IN $\mathbb{C}[-l, l[$ AND AN ESTIMATION OF A USEFUL SIGNAL

There exist many approaches and codes in Matlab which can be used for simulations of various stochastic processes which are described by Ornstein-Uhlenbeck stochastic differential equations(see, for example [3], [7], [5], [10]). Our main attention is devoted to animation and simulation of a useful signal transmitting processes which all are described by the Ornstein-Uhlenbeck stochastic system (1.1). We are going also to demonstrate whether works the statistic T_n constructed in Theorem 3.1. In this context we present some codes in Matlab which are described by the following examples. In all examples we assume that $(\Omega, \mathbb{F}, P) = (\mathbf{R}^N, \mathbb{B}(\mathbf{R}^N), \gamma^N)$, where γ^N denotes N -power of the linear standard Gaussian measure γ in R .

Example 4.1. Below we give an animation of the Ornstein-Uhlenbeck process in $\mathbb{C}[0, \pi[$ which is defined by the following stochastic differential equation

$$d\Psi(t, x, \omega) = 2\Psi(t, x, \omega)dt - 10 \frac{\partial}{\partial x} \Psi(t, x, \omega)dt + 1.174dW(t, \omega), \quad (4.1)$$

with initial condition

$$\Psi(0, x, \omega) = 1/2 + 15\cos x + 3\cos 3x + \cos 8x + 5\sin 3x + 15\sin 5x, \quad (4.2)$$

$$(t, x, \omega) \in [0, \frac{\pi}{7}[\times [-\pi, \pi[\times R^N.$$

```
>> N = 10000;
s = 1.174;
x1 = random('Normal', 0, 1, N, 1000);
A1 = [-10, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0];
C1 = [15, 0, 3, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0];
```

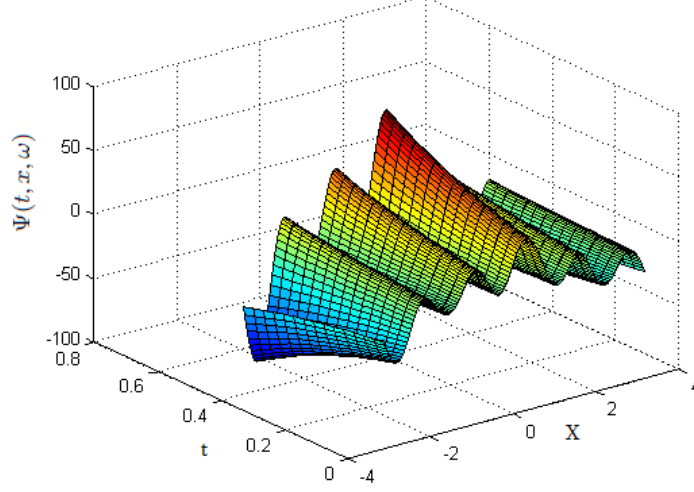


FIGURE 1. Picture from an animation of Ornstein-Uhlenbeck stochastic process defined by stochastic differential equation (4.1) with initial value problem (4.2)

```

D1 = [0, 0, 5, 0, 15, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0];
A10 = 2; A20 = 0; C10 = 1;
for k = 1 : 20
    S1(k) = A10; S2(k) = A20;
    for n = 1 : 10
        S1(k) = S1(k) + (-1)^(n) * A1(2 * n) * k^(2 * n);
    end
end
for k = 1 : 20
    O1(k) = 0;
end
for k = 1 : 20
    for n = 1 : 10
        O1(k) = O1(k) + (-1)^n * A1(2 * n + 1) * k^(2 * n + 1);
    end
end
[T1, X1] = meshgrid(0 : (pi/100) : 2 * pi, -pi : (pi/100) : pi);
for m = 1 : N
    Z1 = 0.5 * C10 * exp(T1 * A10) + s * x1(m, 1) * ((s/sqrt(2 * A10)) * exp(T1 *
    (-1) * A10) * exp(T1 * 2 * A10) - 1);
    for k = 1 : 20
        Z1 = Z1 + C1(k) * exp(T1 * S1(k)) * cos(X1 * k) * cos(T1 * O1(k)) + D1(k) *
        exp(T1 * S1(k)) * cos(X1 * k) * sin(T1 * O1(k)) + D1(k) * exp(T1 * S1(k)) * sin(X1 *
        k) * cos(T1 * O1(k)) -

```



```

C1(k)*exp(T1*S1(k)).*sin(X1.*k).*sin(T1*O1(k))+s*sqrt(2)*x1(m,k+
1)*sin(pi*k*(exp(2*A10*T1)-1))/(pi*k);
end
surf(X1,T1,Z1)
drawnow;
pause(1);
end

```

Example 4.2. Let consider the Ornstein-Uhlenbeck stochastic differential equation

$$d\Psi(t, x, \omega) = 2\Psi(t, x, \omega)dt - \frac{\partial}{\partial x}\Psi(t, x, \omega)dt + \sigma dW(t, \omega), \quad (4.3)$$

with initial condition

$$\Psi(0, x, \omega) = 1/2 + 5\cos x + 5\cos 5x, \quad (4.4)$$

$$(t, x, \omega) \in [0, \frac{\pi}{7}] \times [-\pi, \pi] \times R^\infty.$$

Below we present the programm in Matlab which draw a sample of the size 4 which are results of observations to solutions of the Ornstein-Uhlenbeck stochastic differential equation (4.3) – (4.4) at moment $t = \pi/7$.

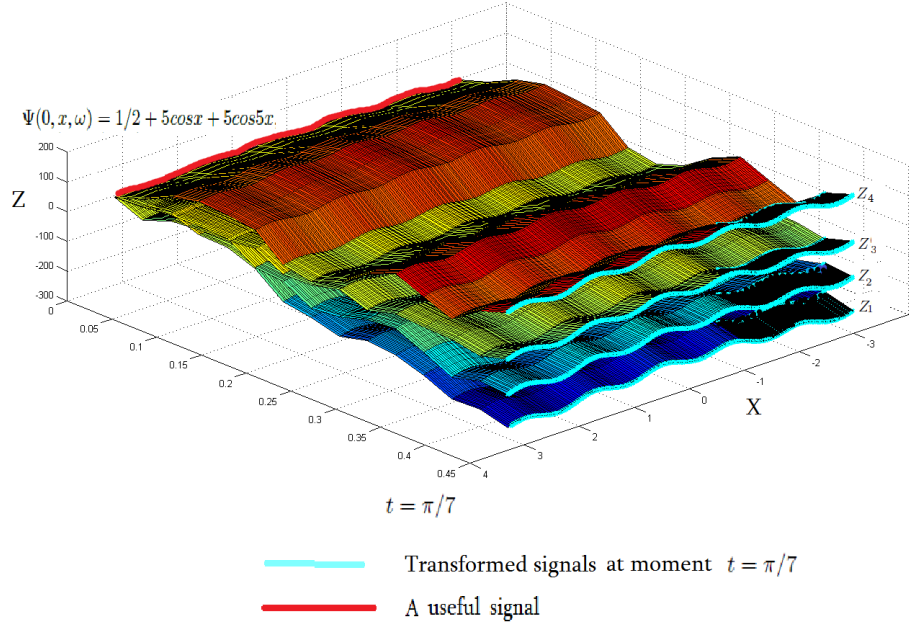


FIGURE 2. A sample of the size 4 which are results of observations to the solutions of the Ornstein-Uhlenbeck stochastic differential equation (4.3)-(4.4) at moment $t = \pi/7$ when $\sigma = 150$

```

>> N = 4;
x1 = random('Normal', 0, 1, N, 1000);
A1 = [-1; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0];

```

```

C1 = [5; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0];
D1 = [0; 0; 0; 0; 5; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0];
A10 = 2; C10 = 1; s = 150;
fork = 1 : 20
S1(k) = A10;
for n = 1 : 10
S1(k) = S1(k) + (-1)(n) * A1(2 * n) * k(2 * n);
end
end
fork = 1 : 20
O1(k) = 0;
end
for k = 1 : 20
for n = 1 : 10
O1(k) = O1(k) + (-1)n * A1(2 * n + 1) * k(2 * n + 1);
end
end
[T1, X1] = meshgrid(0 : (pi/100) : pi/7, -pi : (pi/100) : pi);
form = 1 : N
Zm = 0.5 * C10 * exp(T1 * A10) + (s * x1(1, 1) / sqrt(2 * A10)) * exp(-A10 * T1) *
(exp(2 * A10 * T1) - 1);
end
form = 1 : N
fork = 1 : 20
Zm = Zm + C1(k) * exp(T1 * S1(k)) * cos(X1 * k) * cos(T1 * O1(k)) + D1(k) *
exp(T1 * S1(k)) * cos(X1 * k) * sin(T1 * O1(k)) + D1(k) * exp(T1 * S1(k)) * sin(X1 *
k) * cos(T1 * O1(k)) -
C1(k) * exp(T1 * S1(k)) * sin(X1 * k) * sin(T1 * O1(k)) + (s / (sqrt(2 * A10) * pi *
k)) * sqrt(2) * x1(m, k + 1) * exp(-A10 * T1) * sin(pi * k * (exp(2 * A10 * T1) - 1));
end
end
surf(X1, T1, Z1)
hold on
surf(X2, T2, Z2)
hold on
surf(X3, T3, Z3)
hold on
surf(X4, T4, Z4)
hold off
>>

```

Example 4.3. Suppose that we have a sample $(Z_i)_{1 \leq i \leq n} \in (FD^{(0)}[-\pi, \pi])^n$ of size n . In our simulation, we have that

$$Z_i = (\Psi(t_0, x, \omega^{(i)}))_{x \in [-\pi, \pi]}$$

for $1 \leq i \leq n$, where $\omega^{(i)} = (x_j^{(i)})_{j \in N} \in \mathbf{R}^N$ is γ -uniformly distributed sequence in \mathbf{R} for each $1 \leq i \leq n$. For example, we can put $x_j^{(i)} = \Phi^{-1}(\{j\sqrt{p_i}\})$, where p_i is i -th simple natural number for $i \in N$, $j \in N$, $\{\cdot\}$ denotes a fractal part of the

real number and $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-\frac{y^2}{2}} dy$ for $t \in \mathbb{R}$. In that case we can simulate Wiener trajectory $W(t, (x_j^{(i)})_{j \in N})$ as follows:

$$W(t, (x_j^{(i)})_{j \in N}) = x_0^{(i)} t + \sqrt{2} \sum_{n=1}^{\infty} x_n^{(i)} \frac{\sin \pi n t}{\pi n}. \quad (4.5)$$

Since

$$\begin{aligned} \sigma \int_0^{t_0} e^{(t_0-\tau)\mathbb{A}} dW(\tau, (x_j^{(i)})_{j \in N}) &= \frac{\sigma}{2A_0} e^{-A_0 t_0} (x_0^{(i)} (e^{2A_0 t_0} - 1) + \\ &\quad \sqrt{2} \sum_{n=1}^{\infty} x_n^{(i)} \frac{\sin(\pi n (e^{2A_0 t_0} - 1))}{\pi n}), \end{aligned} \quad (4.6)$$

we can simulate Z_i as follows

$$\begin{aligned} Z_i = (\Psi(t_0, x, (x_j^{(i)})_{j \in N}))_{x \in [-\pi, \pi[} &= \frac{e^{t_0 A_0} c_0}{2} + \sum_{k=1}^{\infty} e^{\sigma_k t_0} \left((c_k \cos(\omega_k t_0) + \right. \\ &\quad \left. d_k \sin(\omega_k t_0)) \cos\left(\frac{k\pi x}{l}\right)_{x \in [-\pi, \pi[} + (d_k \cos(\omega_k t_0) - c_k \sin(\omega_k t_0)) \sin\left(\frac{k\pi x}{l}\right)_{x \in [-\pi, \pi[} \right) + \\ &\quad \frac{\sigma}{2A_0} e^{-A_0 t_0} (x_0^{(i)} (e^{2A_0 t_0} - 1) + \sqrt{2} \sum_{n=1}^{\infty} x_n^{(i)} \frac{\sin(\pi n (e^{2A_0 t_0} - 1))}{\pi n}). \end{aligned} \quad (4.7)$$

Suppose we want to estimate a useful signal $\Psi_0(x)_{x \in [-l, l[} \in FD^{(0)}$ defined by

$$\Psi_0(x)_{x \in [-\pi, \pi[} = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cos\left(\frac{k\pi x}{l}\right)_{x \in [-\pi, \pi[} + d_k \sin\left(\frac{k\pi x}{l}\right)_{x \in [-\pi, \pi[}. \quad (4.8)$$

For a function $f \in FD^{(0)}([-\pi, \pi])$ we put :

$$\tilde{c}_0(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx; \quad (4.9)$$

$$\tilde{c}_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(kx) f(x) dx (k \in \mathbb{N}); \quad (4.10)$$

$$\tilde{d}_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(kx) f(x) dx (k \in \mathbb{N}). \quad (4.11)$$

Suppose that all conditions of Theorem 3.1 are satisfied. Then following Theorem 3.1, an estimator T_n of the useful signal Ψ_0 is given by

$$\begin{aligned} T_n((Z_i)_{1 \leq i \leq n}) &= \frac{e^{-t_0 A_0} \tilde{c}_0(\frac{\sum_{i=1}^n Z_i}{n})}{2} + \sum_{k=1}^{\infty} e^{-\sigma_k t_0} \left((\tilde{c}_k(\frac{\sum_{i=1}^n Z_i}{n}) \cos(\omega_k t_0) + \right. \\ &\quad \tilde{d}_k(\frac{\sum_{i=1}^n Z_i}{n}) \sin(\omega_k t_0)) \cos(kx)_{x \in [-\pi, \pi[} + (\tilde{d}_k(\frac{\sum_{i=1}^n Z_i}{n}) \cos(\omega_k t_0) - \\ &\quad \tilde{c}_k(\frac{\sum_{i=1}^n Z_i}{n}) \sin(\omega_k t_0)) \sin(kx)_{x \in [-\pi, \pi[} \Big) \end{aligned} \quad (4.12)$$

Our next program draws results of calculations of the estimate T_{10} of a useful signal when we have a sample of size 10 which are results of observations to the

solutions of the Ornstein-Uhlenbeck stochastic differential equation (4.3) – (4.4) at moment $t_0 = \pi/7$ and $\sigma \in \{150; 1500; 7500; 15000\}$.

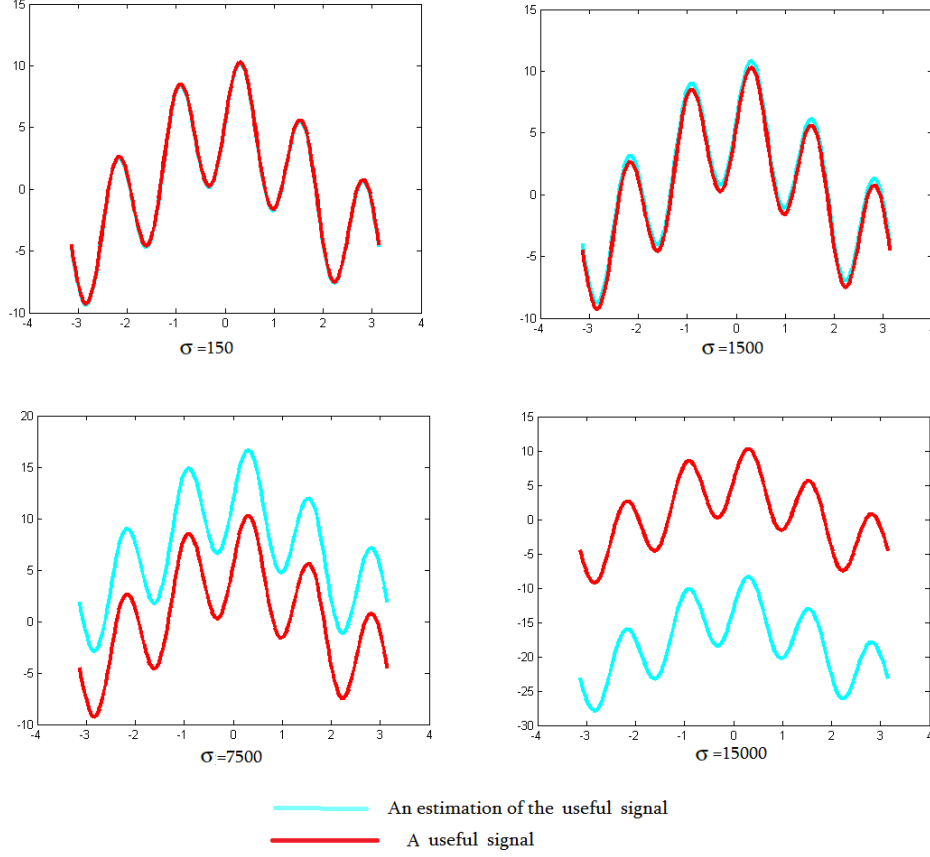


FIGURE 3. An estimation of the useful signal by using the statistic T_{10} in the Ornstein-Uhlenbeck process (4.3)-(4.4) by the sample of the size 10 which are results of observations to transformed signals at moment $t_0 = \pi/7$ when $\sigma \in \{150; 1500; 7500; 15000\}$

```
>> N = 1000; M = 10; s = 150;
x1 = random('Normal', 0, 1, N, 1000);
A1 = [-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0];
C1 = [5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0];
D1 = [0, 0, 0, 0, 5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0];
A10 = 2; C10 = 1;
for k = 1 : 20
    S1(k) = A10; S2(k) = A20;
for n = 1 : 10
    S1(k) = S1(k) + (-1)^n * A1(2 * n) * k^(2*n);
end
```

```

end
for k = 1 : 20
O1(k) = 0;
end
for k = 1 : 20
for n = 1 : 10
O1(k) = O1(k) + (-1)n * A1(2 * n + 1) * k2*n+1;
end
end
T1 = pi/7;
X1 = -pi : (pi/100) : pi;
for m = 1 : M
Zm = 0.5 * C10 * exp(T1 * A10) + 2/sqrt(2 * A10) * exp(-A10 * T1) * x1(m, 1) *
(exp(2 * A10 * T1) - 1);
end
for m = 1 : M
for k = 1 : 20
Zm = Zm + C1(k) * exp(T1 * S1(k)). * cos(X1. * k). * cos(T1 * O1(k)) + D1(k) *
exp(T1 * S1(k)). * cos(X1. * k). * sin(T1 * O1(k)) +
D1(k) * exp(T1 * S1(k)). * sin(X1. * k). * cos(T1 * O1(k)) - C1(k) * exp(T1 *
S1(k)). * sin(X1. * k). * sin(T1 * O1(k)) +
s * 2/sqrt(2 * A10) * exp(-A10 * T1) * sqrt(2) * x1(m, k + 1) * sin(pi * k * (exp(2 *
A10 * T1) - 1))/(pi * k);
end
end
W = 0;
for m = 1 : M
W = W + Zm;
end
W = W/M;
c = 0;
y = W;
for s = 1 : 200
c = c + (2/(2 * pi * 200)) * y(s);
end
for m = 1 : 20
am = 0;
bm = 0;
end
for m = 1 : 20
for k = 1 : 200
am = am + (1/100) * y(k) * cos(m * X1(k));
bm = bm + (1/100) * y(k) * sin(m * X1(k));
end
end
Y = c * exp(-T1 * A10)/2;
for k = 1 : 20

```

```

Y = Y + exp(-S1(k) * T1) * ((a_k * cos(-O1(k) * T1) + b_k * sin(-O1(k) * T1)) *
cos(k * X1) + (b_k * cos(-O1(k) * T1) - a_k * sin(-O1(k) * T1)) * sin(k * X1))
end
Y3 = 5 * cos(X1) + 5 * sin(5 * X1) + C10/2;
plot(X1, Y, 'c', X1, Y3, 'r', 'LineWidth', 3)

```

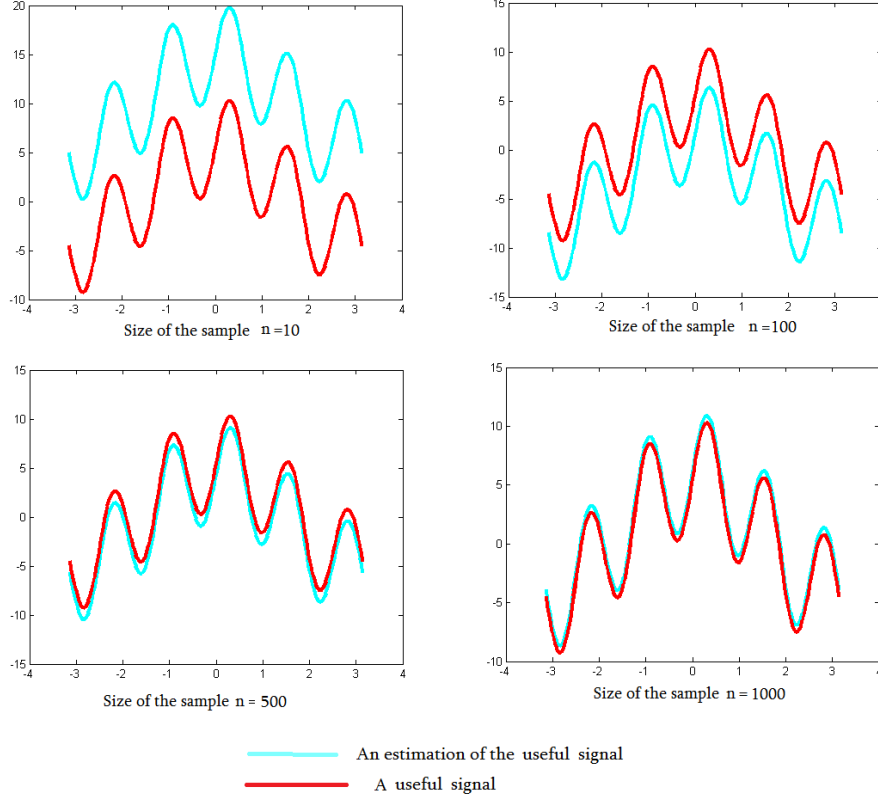


FIGURE 4. An estimation of the useful signal by using the statistic T_n in the Ornstein-Uhlenbeck process (4.5)-(4.6) by the increase of a size of the sample when $\sigma = 15000$

5. DISCUSSION AND CONCLUSION

If a transmitting process of a useful signal $\theta \in \Theta$ is described by the Ornstein-Uhlenbeck stochastic system (1.1) and we have results of observations $(Z_k)_{1 \leq k \leq n}$ on transformed signals at any moment $t_0 > 0$, then following Theorem 3.1, by using the statistic T_n we can restore θ .

Programs in Matlab prepared in the present paper can be described as follows:

(i) A program in Matlab from Example 4.1 demonstrates animation of a particular case of the Ornstein-Uhlenbeck stochastic system (1.1) which are defined by (4.1)-(4.2)(see Figure 1).

(ii) A program in Matlab from Example 4.2 draws and presents a sample $(Z_k)_{1 \leq k \leq n}$ of the size n which consists from results of observations to n independent transformed signals at moment $t = \pi/7$ when a transmitting process of a useful signal θ is described by the Ornstein-Uhlenbeck stochastic system (1.1) defined by (4.3)-(4.4) (see Example 4.2 and Figure 2 for which $n = 4$).

(iii) A program in Matlab from Example 4.3 draws the value of the statistic T_n (which is in $C[-\pi, \pi]$) calculated for sample $(Z_k)_{1 \leq k \leq n}$ of the size n which consists from results of observations to n independent transformed signals at moment $t = \pi/7$ when a transmitting process of a useful signal θ is described by the Ornstein-Uhlenbeck stochastic system (1.1) defined by (4.5)-(4.6). (see Example 4.3 and Figure 3).

From Figure 3 we see that the reduction of the parameter σ in (4.3), for the fixed size of the sample (here, $n = 10$) increases the accuracy of the estimation of the useful signal which seems naturally.

Similarly, from Figure 4 we see that an increase of the size of the sample, for a fixed big value of the parameter σ in (4.3) (here, $\sigma = 1500$), also increases the accuracy of the estimation of the useful signal which do not contradicts to the result of Theorem 3.1.

REFERENCES

1. Gantmacher F. R.: *Theorie des matrices*. Tome 1: Theorie gen- erale. (French) Traduit du Russe par Ch. Sarthou. Collection Universitaire de Mathematiques, No. 18 Dunod, 1966.
2. Garbaczewski, P., Olkiewicz, R.: Ornstein-Uhlenbeck-Cauchy process, J. Math. Phys., 41(2000), 68436860.
3. Gillespie, D. T.: Exact numerical simulation of the Ornstein-Uhlenbeck process and its integral. Physical review E 54, (1996). no. 2: 20842091.
4. Ornstein, L. S., Uhlenbeck, G. E.: On the Theory of the Brownian Motion. Physical Review 36,(1930). no. 5: 823. doi:10.1103/PhysRev.36.823.
5. Labadze, L., Pantsulaia, G.: Estimation of the parameters of the Ornstein-Uhlenbeck's process. <https://arxiv.org/pdf/1608.04507v3.pdf>destination
6. Pantsulaia, G. R., Giorgadze, G.P.: On a Linear Partial Differential Equation of the Higher Order in Two Variables with Initial Condition Whose Coefficients are Real-valued Simple Step Functions, J. Partial Diff. Eqs., 29 (2016) . No. 1, 1-13
7. Labadze, L., Saatashvili, G., Pantsulaia, G.: Infinite-sample consistent estimations of parameters of the Wiener process with drift. <https://arxiv.org/pdf/1611.01119v2.pdf>destination
8. Protter, P.: Stochastic integration and differential equations, Springer-Verlag, Berlin, 2004.
9. Shiryaev, A.N.: *Probability* (in Russian), Izd. "Nauka", Moscow, 1980.
10. Smith, William.: On the Simulation and Estimation of the Mean-Reverting Ornstein-Uhlenbeck Process, Especially as Applied to Commodities Markets and Modelling, Verson 1.01 (February), 2010. <https://commoditymodels.files.wordpress.com/2010/02/estimating-the-parameters-of-a-mean-reverting-ornstein-uhlenbeck-process1.pdf>destination
11. Ibramkhallilov, I.Sh., Skorokhod, A.V.: *On well-off estimates of parameters of stochastic processes* (in Russian), Kiev, 1980.

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